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# Stress analysis of symmetric laminates with obliquely-crossed matrix cracks

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**Abstract**—Simple methodology for stress analysis of laminates with obliquely-crossed matrix cracks is presented. An oblique coordinate system along the matrix cracks is introduced in conjunction with the derivation of oblique coordinate tensor components. Using displacement/strain covariant components and force/stress contravariant components, it is shown that the obliquely crossed crack problem with any oblique angle can be treated as the orthogonally-crossed crack problem, which ensures that obliquely-crossed cracked laminates can be regarded as orthogonally-crossed cracked laminates with appropriate transformation of properties. This approach is combined with two-dimensional shear-lag analysis and analytical solutions of $[\theta_m/90_n]$s and $[S/\theta_m/90_n]$s laminates with matrix cracks in both $\theta$- and 90-plies under general in-plane loadings are obtained in terms of oblique coordinate components. Calculated stress distributions are compared with 3D FEM results and the effectiveness of the present analysis is verified.

*Keywords*: Laminates; matrix cracking; multi-ply damage accumulation; shear-lag analysis.

## 1. INTRODUCTION

Composite laminates are commonly used as various structural components in the aerospace industry. The observed damage process of laminated composites during operation is rather complex consisting of transverse cracks, delaminations, fiber-matrix debondings, fiber fractures, etc. Among them, transverse cracks are often the first observed damage mode under tensile loadings. Although this damage mode is not critical from a final fracture point of view, it induces the reduction of laminate thermo-mechanical performance, more severe damages and leakage of contained liquid or gas. Thus, good understanding of the damage process of

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transverse cracks is necessary for evaluating the performance and durability of composite laminates.

Transverse cracking in the off-axis ply of composite laminates has been extensively studied and several analytical models have been developed to predict effective properties and stress/strain field of laminates with full-width transverse cracks [1–10]. Most of the analytical models, however, are confined only to cross-ply laminates containing central cracked 90-degree layers subjected to uniaxial loadings. Further investigations focused on surface ply cracking [11–13], cross-ply laminates under biaxial and shear loadings [14, 15] have also been analytically performed. For the case of general laminates, it should be noted that analytical models of cross-ply laminates are applicable to balanced symmetric laminates with cracked 90-degree layers in the thickness-wise center provided the constraint layers are not cracked. In addition, accurate analytical model of general symmetric laminates under general in-plane loadings or triaxial loadings is presented by McCartney [16]. These comprehensive researches lead to better understanding and correlation between the predictions and experiments of the damage accumulation in the initially cracked layers that are usually 90-degree layers.

Contrary to the above-mentioned analytical investigations, few analytical works focused on the damage accumulation in the adjacent plies of the cracked layers have been attempted. Some researches deal with effective laminate properties or multilaminar cracking using the damage mechanics approach [17, 18], homogenization method of a cracked ply [19, 20], etc. Although these ‘equivalent’ concepts are tractable and efficient for the prediction of effective mechanical properties, only the effective stresses/strains are obtained and the synergistic effects of the multiply matrix cracks are neglected. For the case of cross-ply laminates with matrix cracks in both 0-degree and 90-degree layers, several analyses including both crack surfaces directly (without using the ‘equivalent’ concept) have been attempted by Hashin [21], Aboudi *et al.* [22], Tsai *et al.* [23, 24], Henaff-Gardin *et al.* [25, 26] and Kashtalyan and Soutis [27]. However, these analytical models are confined only to orthogonally-crossed matrix cracks and extensions to general layups have mathematical difficulties.

In this study, an analytical methodology of obliquely-crossed cracks is presented. Oblique coordinates along crack surfaces are applied to laminates with obliquely-crossed matrix cracks. Covariant displacement/strain and contravariant force/stress components in conjunction with the associated constitutive equation are derived in relation to orthogonal coordinate components. Two-dimensional shear-lag analysis in the oblique coordinates is examined for the case of $[\theta_m/90_n]$s and $[S/\theta_m/90_n]$s laminates with $\theta$-ply and 90-ply matrix cracks using stress boundary conditions in the average sense. Stress distributions of laminates with obliquely-crossed matrix cracks under general in-plane loadings are calculated and verified with the 3D finite element analyses.

## 2. OBLIQUE COORDINATE SYSTEM AND GOVERNING EQUATIONS

### 2.1. Oblique coordinate system

Let $x$ and $y$ denote coordinates along reference lines $r_1$ and $r_2$, which are envisioned as the obliquely-crossed crack surfaces as shown in Fig. 1a, with the coordinate $z$ along the line perpendicular to the plane including the lines $r_1$ and $r_2$. The covariant base vectors $\mathbf{g}_i$ with unit length are set along these coordinate lines as shown in Fig. 1b, while the contravariant base vectors $\mathbf{g}^i$ can be defined with the reciprocal relationship between these vectors. An arbitrary vector can be resolved along the covariant and contravariant base vectors as [28]

$$\mathbf{b} = b^i \mathbf{g}_i = b_i \mathbf{g}^i, \tag{1}$$

where $b_i$ and $b^i$ denote the covariant and contravariant components of the vector. Also, a second-order tensor $\mathbf{A}$ is expressed as

$$\mathbf{A} = A^{ij} \mathbf{g}_i \otimes \mathbf{g}_j = A^i_{\ j} \mathbf{g}_i \otimes \mathbf{g}^j = A_i^{\ j} \mathbf{g}^i \otimes \mathbf{g}_j = A_{ij} \mathbf{g}^i \otimes \mathbf{g}^j, \tag{2}$$

where $A_{ij}$ and $A^{ij}$ are covariant and contravariant components and $A_i^{\ j}$ and $A^i_{\ j}$ denote mixed components of the tensor.

Considering that ordinary analysis in solid mechanics is performed in the orthogonal coordinates, the $\bar{x}$–$\bar{y}$–$\bar{z}$ orthogonal coordinate system, based on orthogonal unit base vectors $\mathbf{i}_1$, $\mathbf{i}_2$, and $\mathbf{i}_3$, is set in a manner that the line of $\bar{x}$ is arranged at $\theta$-degrees to the line $r_1$ and $\bar{y}$, $\bar{z}$ are located in the same directions as $y$, $z$ as shown in Fig. 1a. The coordinate transformation of a vector $\mathbf{b}$ from $\bar{x}$–$\bar{y}$–$\bar{z}$ system to $x$–$y$–$z$ system is expressed in the matrix form, for the covariant components, as

$$[b_i] = \begin{bmatrix} \cos\theta & \sin\theta & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} [\bar{b}] \equiv [Z][\bar{b}], \tag{3}$$

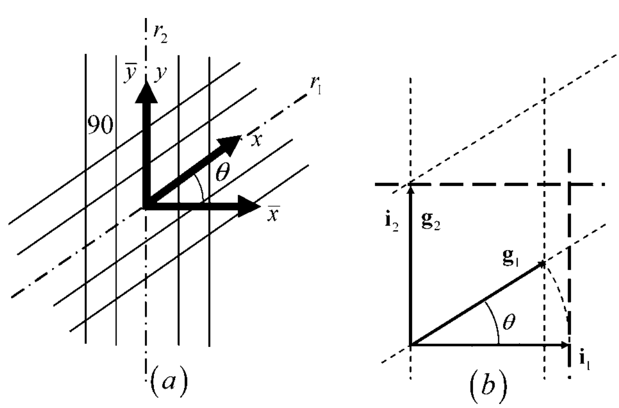



**Figure 1.** Orthogonal and oblique coordinate system: (a) coordinate system, (b) base vector.

where $[b_i] = [b_1\ b_2\ b_3]^T$, etc., and for the contravariant components

$$[b^i] = [Z]^{-T}[\bar{b}]. \tag{4}$$

Using these transformation matrices, covariant components of the second-order tensors can be obtained in the oblique coordinates,

$$[A_{ij}] = [Z][\bar{A}][Z]^T, \tag{5}$$

and the contravariant components are derived as

$$[A^{ij}] = [Z]^{-T}[\bar{A}][Z]^{-1}. \tag{6}$$

In the following, the repeated coordinate indices are reduced in matrix-form notation, e.g. $\sigma_{xx} \rightarrow \sigma_x$. As discussed later, the covariant components of displacement/strain and the contravariant force/stress components are used in this study. The relationship between the corresponding oblique coordinate components and orthogonal coordinate components are derived straightforward using equations (3)–(6) as

Displacement (covariant): $[u_i] = [u, v, w]^T$

$$\begin{bmatrix} u \\ v \\ w \end{bmatrix} = [Z] \begin{bmatrix} \bar{u} \\ \bar{v} \\ \bar{w} \end{bmatrix}. \tag{7}$$

Force (contravariant): $[F^i] = [F^x, F^y, F^z]^T$

$$\begin{bmatrix} F^x \\ F^y \\ F^z \end{bmatrix} = [Z]^{-T} \begin{bmatrix} \bar{F}_x \\ \bar{F}_y \\ \bar{F}_z \end{bmatrix}. \tag{8}$$

Strain (covariant): $[\varepsilon_{ij}] = [\varepsilon_x, \varepsilon_y, \varepsilon_z, \gamma_{yz}, \gamma_{xz}, \gamma_{xy}]^T$

$$\begin{bmatrix} \varepsilon_x \\ \varepsilon_y \\ \varepsilon_z \\ \gamma_{yz} \\ \gamma_{xz} \\ \gamma_{xy} \end{bmatrix} = \begin{bmatrix} \cos^2\theta & \sin^2\theta & 0 & 0 & 0 & \cos\theta\sin\theta \\ 0 & 1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & \sin\theta & \cos\theta & 0 \\ 0 & 2\sin\theta & 0 & 0 & 0 & \cos\theta \end{bmatrix} \begin{bmatrix} \bar{\varepsilon}_x \\ \bar{\varepsilon}_y \\ \bar{\varepsilon}_z \\ \bar{\gamma}_{yz} \\ \bar{\gamma}_{xz} \\ \bar{\gamma}_{xy} \end{bmatrix} \equiv [T][\bar{\varepsilon}]. \tag{9}$$

Stress (contravariant): $[\sigma^{ij}] = [\sigma^x, \sigma^y, \sigma^z, \tau^{yz}, \tau^{xz}, \tau^{xy}]^T$

$$\begin{bmatrix} \sigma^x \\ \sigma^y \\ \sigma^z \\ \tau^{yz} \\ \tau^{xz} \\ \tau^{xy} \end{bmatrix} = [T]^{-T} \begin{bmatrix} \bar{\sigma}_x \\ \bar{\sigma}_y \\ \bar{\sigma}_z \\ \bar{\tau}_{yz} \\ \bar{\tau}_{xz} \\ \bar{\tau}_{xy} \end{bmatrix}. \tag{10}$$

### 2.2. Governing equation

In the non-orthogonal coordinate system, stress-strain or force-displacement components should be set as contravariant–covariant, mixed–mixed, or covariant–contravariant pairs to obtain physically significant work or energy [28]. Considering the symmetry characteristics, displacement-strain relationship and Cauchy's formula, it is natural to apply displacement/strain covariant components and force/stress contravariant components to the analysis in non-orthogonal coordinates. Therefore, the above-mentioned components are applied in this study.

The equilibrium equations in terms of oblique coordinate components can be written as

$$\sigma^{ij}|_j + X^i = 0, \tag{11}$$

where $X^i$ is the contravariant component of body force and $|_j$ stands for the covariant derivative. The Cauchy's formula is expressed as

$$F^i = \sigma^{ji} n_j, \tag{12}$$

$F^i$ and $n_i$ denote the contravariant components of force vector and covariant components of the normal vector respectively. Under infinitesimal strain assumption, the displacement-strain relationship can be expressed as

$$\varepsilon_{ij} = \frac{1}{2}(u_i|_j + u_j|_i). \tag{13}$$

The constitutive equations between the contravariant stress components and the covariant strain components are derived including initial strains $\boldsymbol{\varepsilon}^0$ as

$$\sigma^{ij} = Q^{ijkl}(\varepsilon_{kl} - \varepsilon^0_{kl}), \tag{14}$$

where the stiffness components (4th order contravariant) can be obtained using equations (9) and (10) as

$$[Q^{ijkl}] = [T]^{-T}[\bar{Q}][T]^{-1}. \tag{15}$$

Each component is given in the Appendix.

When governing equations are expressed in terms of oblique coordinate components, the obliquely-crossed crack problem with arbitrary oblique angles can be regarded as an orthogonally-crossed crack problem as depicted in Fig. 2. In real problems, laminates with obliquely-crossed crack geometry are considered. However, in the analytical procedure, transformation to models with orthogonal crack geometry, in which oblique coordinate components should be applied, is valid. Therefore, analysis of the obliquely-crossed problems can be performed using the orthogonally-crossed models with appropriate oblique coordinate vector/tensor components. The finite element method or any other analytical methods may be combined with the present scheme.

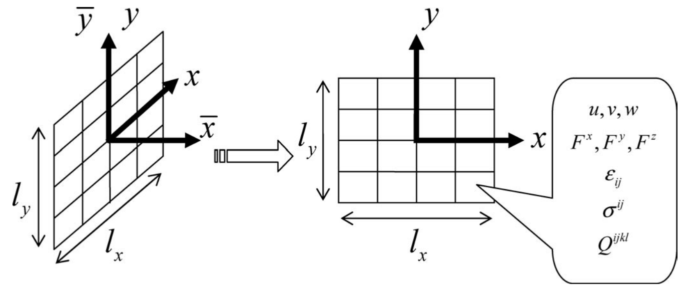



**Figure 2.** Transformation of obliquely-crossed problem to orthogonally-crossed problem with oblique coordinate components.



## 3. STRESS ANALYSIS OF LAMINATES WITH OBLIQUELY CROSSED CRACKS

Stress analytical methods of [$\theta_{\mathrm{m}}$/90$_{\mathrm{n}}$]s and [S/$\theta_{\mathrm{m}}$/90$_{\mathrm{n}}$]s laminates with matrix cracks in both $\theta$- and 90-plies (S denotes sublaminate) under general in-plane loadings are derived using the above-mentioned oblique coordinate components combined with two-dimensional shear-lag model. Although [$\phi_{\mathrm{m}}$/$\varphi_{\mathrm{n}}$]s and [S$'$/$\phi_{\mathrm{m}}$/$\varphi_{\mathrm{n}}$]s laminates with matrix cracks in both $\phi$- and $\varphi$-plies should be generally considered, these layups can be regarded as [$\theta_{\mathrm{m}}$/90$_{\mathrm{n}}$]s and [S/$\theta_{\mathrm{m}}$/90$_{\mathrm{n}}$]s laminates after conventional rotational transformation. Thus, the oblique coordinate system defined in Section 2.1 is applied to the [$\theta_{\mathrm{m}}$/90$_{\mathrm{n}}$]s and [S/$\theta_{\mathrm{m}}$/90$_{\mathrm{n}}$]s laminates in a manner that $x$–$y$ coordinates coincide with the fiber directions of $\theta$- and 90-plies, respectively. The analytical solutions are obtained by solving the governing equations with boundary conditions using the orthogonally-crossed models incorporating the oblique coordinate components.

### *3.1. Problem set*

For the modeling of [$\theta_{\mathrm{m}}$/90$_{\mathrm{n}}$]s and [S/$\theta_{\mathrm{m}}$/90$_{\mathrm{n}}$]s laminates with matrix cracks in both $\theta$- and 90-plies, it is assumed that matrix cracks in both layers exist with uniform densities completely traversing in the fiber direction and are symmetrically arranged with respect to the central plane. The crack densities in $\theta$- and 90-degree plies are set as $1/(2l_1 \cos\theta)$ and $1/(2l_2 \cos\theta)$ respectively, where $l_1$ and $l_2$ are half-lengths of crack spacing in $\theta$- and 90-degree plies along the oblique coordinate system. Thus, the representative volume to be analyzed can be taken as shown in Fig. 3 for (a) [$\theta_{\mathrm{m}}$/90$_{\mathrm{n}}$]s and (b) [S/$\theta_{\mathrm{m}}$/90$_{\mathrm{n}}$]s. The obliquely-crossed problem can be regarded as an orthogonally-crossed problem with appropriate components as shown in Fig. 3. The oblique components of applied in-plane stresses $[\sigma] = [\sigma^x\ \sigma^y\ \tau^{xy}]^T$ or strains $[\varepsilon] = [\varepsilon_x\ \varepsilon_y\ \gamma_{xy}]^T$ can be calculated using the orthogonal components $[\bar{\sigma}] = [\bar{\sigma}_x\ \bar{\sigma}_y\ \bar{\tau}_{xy}]^T$ or $[\bar{\varepsilon}] = [\bar{\varepsilon}_x\ \bar{\varepsilon}_y\ \bar{\gamma}_{xy}]^T$ and equations (10) or (9). In notation, ply indices of S-, $\theta$- and 90-plies corresponds to 0, 1 and 2 respectively inscribed in superscript parentheses and superscript $(k, k+1)$ means the interface between $k$-th ply and $(k+1)$-th ply.

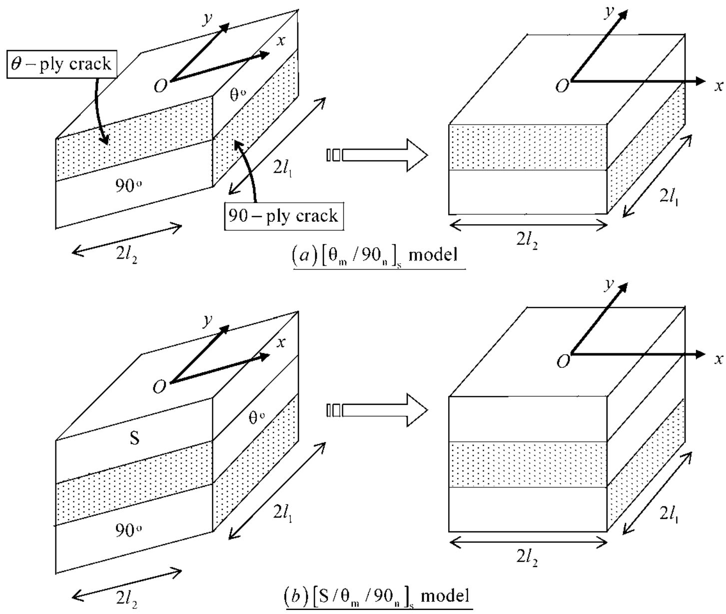



**Figure 3.** $[\theta_{\mathrm{m}}/90_{\mathrm{n}}]$s and $[\mathrm{S}/\theta_{\mathrm{m}}/90_{\mathrm{n}}]$s laminate models with obliquely-crossed matrix cracks in $\theta$- and 90-plies: (a) representative volume in real problem, (b) transformation to orthogonal problem.

### *3.2. Shear lag model*

In the shear-lag model, only in-plane stresses in each ply and out-of-plane shear stresses at the interfaces between the plies are incorporated. The in-plane stresses are assumed to be constant in the thickness direction, or through-the-thickness averaged stresses are taken as generalized in-plane stresses. In this study, constitutive equations between in-plane displacements in each ply and out-of-plane shear stresses at the interfaces, which were derived by Caron and Carreira [29] using the Hellinger-Reissner variational principle, are utilized

$$\begin{bmatrix} \bar{u}^{(k)} - \bar{u}^{(k+1)} \\ \bar{v}^{(k)} - \bar{v}^{(k+1)} \end{bmatrix} = [\bar{I}^{(k)}] \begin{bmatrix} \bar{\tau}_{xz}^{(k-1,k)} \\ \bar{\tau}_{yz}^{(k-1,k)} \end{bmatrix} + [\bar{C}^{(k,k+1)}] \begin{bmatrix} \bar{\tau}_{xz}^{(k,k+1)} \\ \bar{\tau}_{yz}^{(k,k+1)} \end{bmatrix} + [\bar{I}^{(k+1)}] \begin{bmatrix} \bar{\tau}_{xz}^{(k+1,k+2)} \\ \bar{\tau}_{yz}^{(k+1,k+2)} \end{bmatrix}, \tag{16}$$

where

$$[\bar{C}^{(k,k+1)}] = \frac{t^{(k)}}{3} \begin{bmatrix} \bar{S}_{55}^{(k)} & \bar{S}_{45}^{(k)} \\ \bar{S}_{45}^{(k)} & \bar{S}_{44}^{(k)} \end{bmatrix} + \frac{t^{(k+1)}}{3} \begin{bmatrix} \bar{S}_{55}^{(k+1)} & \bar{S}_{45}^{(k+1)} \\ \bar{S}_{45}^{(k+1)} & \bar{S}_{44}^{(k+1)} \end{bmatrix},$$
$$[\bar{I}^{(k)}] = \frac{t^{(k)}}{6} \begin{bmatrix} \bar{S}_{55}^{(k)} & \bar{S}_{45}^{(k)} \\ \bar{S}_{45}^{(k)} & \bar{S}_{44}^{(k)} \end{bmatrix}, \tag{17}$$

and $S_{ij}$ and $t$ denote the compliance and thickness. It should be noted that equations (16) and (17) are expressed in terms of orthogonal coordinate components. With the conditions of stress-free surface and central surface, interface constitutive equations of [$\theta_m$/$90_n$]s laminates can be derived as

$$\begin{bmatrix} \bar{\tau}_{xz}^{(1,2)} \\ \bar{\tau}_{yz}^{(1,2)} \end{bmatrix} = [\bar{C}^{(1,2)}]^{-1} \begin{bmatrix} \bar{u}^{(1)} - \bar{u}^{(2)} \\ \bar{v}^{(1)} - \bar{v}^{(2)} \end{bmatrix} \equiv [\bar{H}] \begin{bmatrix} \bar{u}^{(1)} - \bar{u}^{(2)} \\ \bar{v}^{(1)} - \bar{v}^{(2)} \end{bmatrix}. \tag{18}$$

Transformation to the oblique coordinate components leads to

$$\begin{aligned} \begin{bmatrix} \tau^{xz(1,2)} \\ \tau^{yz(1,2)} \end{bmatrix} &= \begin{bmatrix} \sec\theta & 0 \\ -\tan\theta & 1 \end{bmatrix} [\bar{H}] \begin{bmatrix} \cos\theta & \sin\theta \\ 0 & 1 \end{bmatrix}^{-1} \begin{bmatrix} u^{(1)} - u^{(2)} \\ v^{(1)} - v^{(2)} \end{bmatrix} \\ &\equiv [H] \begin{bmatrix} u^{(1)} - u^{(2)} \\ v^{(1)} - v^{(2)} \end{bmatrix}. \end{aligned} \tag{19}$$

Similarly, interface constitutive equations of [S/$\theta_m$/$90_n$]s laminates are expressed in terms of oblique components as

$$\begin{aligned} \begin{bmatrix} \tau^{xz(0,1)} \\ \tau^{yz(0,1)} \\ \tau^{xz(1,2)} \\ \tau^{yz(1,2)} \end{bmatrix} &= \begin{bmatrix} C_{11}^{(0,1)} & C_{12}^{(0,1)} & I_{11}^{(1)} & I_{12}^{(1)} \\ C_{21}^{(0,1)} & C_{22}^{(0,1)} & I_{21}^{(1)} & I_{22}^{(1)} \\ I_{11}^{(1)} & I_{12}^{(1)} & C_{11}^{(1,2)} & C_{12}^{(1,2)} \\ I_{21}^{(1)} & I_{22}^{(1)} & C_{21}^{(1,2)} & C_{22}^{(1,2)} \end{bmatrix} \begin{bmatrix} u^{(0)} - u^{(1)} \\ v^{(0)} - v^{(1)} \\ u^{(1)} - u^{(2)} \\ v^{(1)} - v^{(2)} \end{bmatrix} \\ &\equiv [H] \begin{bmatrix} u^{(0)} - u^{(1)} \\ v^{(0)} - v^{(1)} \\ u^{(1)} - u^{(2)} \\ v^{(1)} - v^{(2)} \end{bmatrix}, \end{aligned} \tag{20}$$

where

$$\begin{aligned} [C^{(k,k+1)}] &= \begin{bmatrix} \cos\theta & \sin\theta \\ 0 & 1 \end{bmatrix} [\bar{C}^{(k,k+1)}] \begin{bmatrix} \sec\theta & 0 \\ -\tan\theta & 1 \end{bmatrix}^{-1}, \\ [I^{(k)}] &= \begin{bmatrix} \cos\theta & \sin\theta \\ 0 & 1 \end{bmatrix} [\bar{I}^{(k)}] \begin{bmatrix} \sec\theta & 0 \\ -\tan\theta & 1 \end{bmatrix}^{-1}. \end{aligned} \tag{21}$$

Equations (19) and (20) are interface constitutive equations based on oblique components for [$\theta_m$/$90_n$]s and [S/$\theta_m$/$90_n$]s laminates respectively.

### 3.3. Stress analysis of [$\theta_m$/$90_n$]s laminates

Through-the-thickness integrated equilibrium equations can be expressed using (11) as

$$\begin{aligned} \tau^{xz(1,2)} &= t^{(1)}\frac{\partial \sigma^{x(1)}}{\partial x} + t^{(1)}\frac{\partial \tau^{xy(1)}}{\partial y} = -t^{(2)}\frac{\partial \sigma^{x(2)}}{\partial x} - t^{(2)}\frac{\partial \tau^{xy(2)}}{\partial y}, \\ \tau^{yz(1,2)} &= t^{(1)}\frac{\partial \sigma^{y(1)}}{\partial y} + t^{(1)}\frac{\partial \tau^{xy(1)}}{\partial x} = -t^{(2)}\frac{\partial \sigma^{y(2)}}{\partial y} - t^{(2)}\frac{\partial \tau^{xy(2)}}{\partial x}. \end{aligned} \tag{22}$$

Combination of (13), (14), (19) and (22) leads to

$$\begin{aligned} \Delta_1^{(1)} u^{(1)} + \Delta_2^{(1)} v^{(1)} &= H_{11}(u^{(1)} - u^{(2)}) + H_{12}(v^{(1)} - v^{(2)}), \\ \Delta_2^{(1)} u^{(1)} + \Delta_3^{(1)} v^{(1)} &= H_{21}(u^{(1)} - u^{(2)}) + H_{22}(v^{(1)} - v^{(2)}), \\ \Delta_1^{(2)} u^{(2)} + \Delta_2^{(2)} v^{(2)} &= H_{11}(u^{(2)} - u^{(1)}) + H_{12}(v^{(2)} - v^{(1)}), \\ \Delta_2^{(2)} u^{(2)} + \Delta_3^{(2)} v^{(2)} &= H_{21}(u^{(2)} - u^{(1)}) + H_{22}(v^{(2)} - v^{(1)}), \end{aligned} \tag{23}$$

where

$$\begin{aligned} \Delta_1^{(k)} &= A_{11}^{(k)}\frac{\partial^2}{\partial x^2} + 2A_{16}^{(k)}\frac{\partial^2}{\partial x \partial y} + A_{66}^{(k)}\frac{\partial^2}{\partial y^2}, \\ \Delta_2^{(k)} &= A_{16}^{(k)}\frac{\partial^2}{\partial x^2} + (A_{12}^{(k)} + A_{66}^{(k)})\frac{\partial^2}{\partial x \partial y} + A_{26}^{(k)}\frac{\partial^2}{\partial y^2}, \\ \Delta_3^{(k)} &= A_{66}^{(k)}\frac{\partial^2}{\partial x^2} + 2A_{26}^{(k)}\frac{\partial^2}{\partial x \partial y} + A_{22}^{(k)}\frac{\partial^2}{\partial y^2}, \end{aligned} \tag{24}$$

$$A_{ij}^{(k)} = t^{(k)} Q_{ij}^{(k)}, \tag{25}$$

and $H_{ij}$ and $Q_{ij}$ denote the components of the matrix $[H]$ in equation (19) and in-plane properties of $[Q^{ijkl}]$ in equation (15) respectively.

To solve the equations (23), the following solutions are assumed

$$\begin{bmatrix} u^{(1)} \\ u^{(2)} \\ v^{(1)} \\ v^{(2)} \end{bmatrix} = \begin{bmatrix} a_1 \\ a_2 \\ a_3 \\ a_4 \end{bmatrix} \sinh(\lambda x) + \begin{bmatrix} b_1 \\ b_2 \\ b_3 \\ b_4 \end{bmatrix} \sinh(\kappa y) + \begin{bmatrix} \varepsilon_x^P \\ \varepsilon_x^P \\ \frac{1}{2}\gamma_{xy}^P \\ \frac{1}{2}\gamma_{xy}^P \end{bmatrix} x + \begin{bmatrix} \frac{1}{2}\gamma_{xy}^P \\ \frac{1}{2}\gamma_{xy}^P \\ \varepsilon_y^P \\ \varepsilon_y^P \end{bmatrix} y, \tag{26}$$

where $[\varepsilon^p] = [\varepsilon_x^p, \varepsilon_y^p, \gamma_{xy}^p]^T$ are unknown constants. Substitution of (26) into (23) indicates that the general solution can be obtained as an eigenvalue problem. The two positive real sets of characteristic values $\lambda_i$ and $\kappa_i$ $(i = 1, 2)$ and corresponding vectors $[a_i] = [a_{i1}\ a_{i2}\ a_{i3}\ a_{i4}]^T$ and $[b_i] = [b_{i1}\ b_{i2}\ b_{i3}\ b_{i4}]^T$ are obtained for the common material properties.

The covariant strain components can be expressed from (13) as

$$\begin{bmatrix} \varepsilon_x^{(k)} \\ \varepsilon_y^{(k)} \\ \gamma_{xy}^{(k)} \end{bmatrix} = \sum_{i=1}^{2} A_i\big[m_i^{(k)}\big]\cosh(\lambda_i x) + \sum_{i=1}^{2} B_i\big[n_i^{(k)}\big]\cosh(\kappa_i y) + [\varepsilon^p], \tag{27}$$

and contravariant stress components are derived as

$$\begin{bmatrix} \sigma^{x(k)} \\ \sigma^{y(k)} \\ \tau^{xy(k)} \end{bmatrix} = \sum_{i=1}^{2} A_i\big[g_i^{(k)}\big]\cosh(\lambda_i x) + \sum_{i=1}^{2} B_i\big[h_i^{(k)}\big]\cosh(\kappa_i y) + [\sigma^{p(k)}], \tag{28}$$

where $A_i$ and $B_i$ are unknown constants and,

$$\begin{aligned} \big[m_i^{(1)}\big] &= [a_{i1}\lambda_i \quad 0 \quad a_{i3}\lambda_i]^T,\ \big[m_i^{(2)}\big] = [a_{i2}\lambda_i \quad 0 \quad a_{i4}\lambda_i]^T, \\ \big[n_i^{(1)}\big] &= [0 \quad b_{i3}\kappa_i \quad b_{i1}\kappa_i]^T,\ \big[n_i^{(2)}\big] = [0 \quad b_{i4}\kappa_i \quad b_{i2}\kappa_i]^T, \end{aligned} \tag{29}$$

$$\begin{aligned} \big[g_i^{(k)}\big] &= [Q^{(k)}]\big[m_i^{(k)}\big], \\ \big[h_i^{(k)}\big] &= [Q^{(k)}]\big[n_i^{(k)}\big], \end{aligned} \tag{30}$$

$$\big[\sigma^{p(k)}\big] = \big[Q^{(k)}\big]\big[\varepsilon^p - \varepsilon^{T(k)}\big]. \tag{31}$$

Note that $[Q^{(k)}]$ and $\varepsilon^{T(k)}$ are in-plane stiffness and thermal strains in the $k$-th ply, respectively.

The boundary conditions are used in average sense as

$$x = \pm l_2,\ \begin{cases} \dfrac{1}{2l_1}\displaystyle\int_{-l_1}^{l_1} \sigma^{x(2)}\,\mathrm{d}y = 0, & \dfrac{1}{2l_1}\displaystyle\int_{-l_1}^{l_1} \tau^{xy(2)}\,\mathrm{d}y = 0, \\ \dfrac{1}{2l_1}\displaystyle\int_{-l_1}^{l_1} \sigma^{x(1)}\,\mathrm{d}y = \dfrac{t^{(1)}+t^{(2)}}{t^{(1)}}\sigma^x, & \dfrac{1}{2l_1}\displaystyle\int_{-l_1}^{l_1} \tau^{xy(1)}\,\mathrm{d}y = \dfrac{t^{(1)}+t^{(2)}}{t^{(1)}}\tau^{xy}, \end{cases} \tag{32}$$

$$y = \pm l_1,\ \begin{cases} \dfrac{1}{2l_2}\displaystyle\int_{-l_2}^{l_2} \sigma^{y(1)}\,\mathrm{d}x = 0, & \dfrac{1}{2l_2}\displaystyle\int_{-l_2}^{l_2} \tau^{xy(1)}\,\mathrm{d}x = 0, \\ \dfrac{1}{2l_2}\displaystyle\int_{-l_2}^{l_2} \sigma^{y(2)}\,\mathrm{d}x = \dfrac{t^{(1)}+t^{(2)}}{t^{(2)}}\sigma^y, & \dfrac{1}{2l_2}\displaystyle\int_{-l_2}^{l_2} \tau^{xy(2)}\,\mathrm{d}x = \dfrac{t^{(1)}+t^{(2)}}{t^{(2)}}\tau^{xy}. \end{cases} \tag{33}$$

It should be noted that the last equation in (33) is satisfied under the requirement of the other equations. Thus, unknown constants $A_1$, $A_2$, $B_1$, $B_2$, $\varepsilon_x^p$, $\varepsilon_y^p$, $\gamma_{xy}^p$ can be determined using (32) and (33). The strain/stress components in the orthogonal coordinate system can be obtained using the inverse relationship of (9) and (10) as functions of oblique coordinates $x$ and $y$.

*3.4. Stress analysis of [S/$\theta_{\mathrm{m}}$/$90_{\mathrm{n}}$]s laminates*

The analytical procedure is identical with the previous section. Through-the-thickness integrated equilibrium equations are expressed as

$$\begin{aligned}
\tau^{xz(0,1)} &= t^{(0)}\frac{\partial\sigma^{x(0)}}{\partial x} + t^{(0)}\frac{\partial\tau^{xy(0)}}{\partial y},\\
\tau^{yz(0,1)} &= t^{(0)}\frac{\partial\sigma^{y(0)}}{\partial y} + t^{(0)}\frac{\partial\tau^{xy(0)}}{\partial x},\\
\tau^{xz(1,2)} - \tau^{xz(0,1)} &= t^{(1)}\frac{\partial\sigma^{x(1)}}{\partial x} + t^{(1)}\frac{\partial\tau^{xy(1)}}{\partial y},\\
\tau^{yz(1,2)} - \tau^{yz(0,1)} &= t^{(1)}\frac{\partial\sigma^{y(1)}}{\partial y} + t^{(1)}\frac{\partial\tau^{xy(1)}}{\partial x},\\
-\tau^{xz(1,2)} &= t^{(2)}\frac{\partial\sigma^{x(2)}}{\partial x} + t^{(2)}\frac{\partial\tau^{xy(2)}}{\partial y},\\
-\tau^{yz(1,2)} &= t^{(2)}\frac{\partial\sigma^{y(2)}}{\partial y} + t^{(2)}\frac{\partial\tau^{xy(2)}}{\partial x}.
\end{aligned} \tag{34}$$

Combination of equations (13), (14), (20) and (34) leads to the simultaneous differential equations. Following the same procedure as the case of [$\theta_{\mathrm{m}}$/$90_{\mathrm{n}}$]s laminates, the four positive real sets of characteristic values $\lambda_i$ and $\kappa_i$ ($i = 1, 2, 3, 4$) and corresponding vectors $[a_i]$ and $[b_i]$ are obtained.

The covariant strain components can be expressed as

$$[\varepsilon^{(k)}] = \sum_{i=1}^{4} A_i\left[m_i^{(k)}\right]\cosh(\lambda_i x) + \sum_{i=1}^{4} B_i\left[n_i^{(k)}\right]\cosh(\kappa_i y) + [\varepsilon^{p}], \tag{35}$$

and contravariant stress components are derived as

$$[\sigma^{(k)}] = \sum_{i=1}^{4} A_i\left[g_i^{(k)}\right]\cosh(\lambda_i x) + \sum_{i=1}^{4} B_i\left[h_i^{(k)}\right]\cosh(\kappa_i y) + [\sigma^{p(k)}], \tag{36}$$

where

$$\begin{aligned}
\left[m_i^{(0)}\right] &= [a_{i1}\lambda_i \quad 0 \quad a_{i4}\lambda_i]^T, \quad \left[m_i^{(1)}\right] = [a_{i2}\lambda_i \quad 0 \quad a_{i5}\lambda_i]^T,\\
\left[m_i^{(2)}\right] &= [a_{i3}\lambda_i \quad 0 \quad a_{i6}\lambda_i]^T, \quad \left[n_i^{(0)}\right] = [0 \quad b_{i4}\kappa_i \quad b_{i1}\kappa_i]^T,\\
\left[n_i^{(1)}\right] &= [0 \quad b_{i5}\kappa_i \quad b_{i2}\kappa_i]^T, \quad \left[n_i^{(2)}\right] = [0 \quad b_{i6}\kappa_i \quad b_{i3}\kappa_i]^T
\end{aligned} \tag{37}$$

are utilized along with equations (30) and (31).

The boundary conditions applied to this problem are similar to the previous case. When in-plane strains (covariant components) are prescribed, the average strains in S-ply are identical to the whole laminate strains. Thus,

$$\frac{1}{4l_1l_2}\int_{-l_1}^{l_1}\int_{-l_2}^{l_2}\varepsilon_{ij}^{(0)}\,\mathrm{d}x\,\mathrm{d}y = \varepsilon_{ij} \tag{38}$$

are used and iso-displacement conditions on the crack plane except for crack surfaces lead to

$$x = \pm l_2, \begin{cases} \int_0^{l_1} (u^{(0)} - v^{(0)} \sin\theta)\,\mathrm{d}y = \int_0^{l_1} (u^{(1)} - v^{(1)} \sin\theta)\,\mathrm{d}y \\ \int_{-l_1}^{0} (u^{(0)} - v^{(0)} \sin\theta)\,\mathrm{d}y = \int_{-l_1}^{0} (u^{(1)} - v^{(1)} \sin\theta)\,\mathrm{d}y, \end{cases} \tag{39}$$

$$y = \pm l_1, \begin{cases} \int_0^{l_2} (v^{(0)} - u^{(0)} \sin\theta)\,\mathrm{d}x = \int_0^{l_2} (v^{(2)} - u^{(2)} \sin\theta)\,\mathrm{d}x \\ \int_{-l_2}^{0} (v^{(0)} - u^{(0)} \sin\theta)\,\mathrm{d}x = \int_{-l_2}^{0} (v^{(2)} - u^{(2)} \sin\theta)\,\mathrm{d}x, \end{cases} \tag{40}$$

Combined with the stress-free conditions on crack surfaces, unknown constants can be determined. Therefore strain/stress distributions in [S/$\theta_m$/$90_n$]s laminates with matrix cracks in both $\theta$- and 90-plies under general in-plane loadings are obtained analytically.

## 4. ANALYTICAL RESULTS AND DISCUSSIONS

In order to verify the effectiveness of the present analytical method, stress distributions of laminates containing obliquely-crossed matrix cracks are calculated and compared with the 3D finite element analysis.

[$\theta$/90]s laminates are considered first. The ply thickness is set as 0.125 [mm], crack spacings are assumed as $l_1 = 0.50$ [mm], $l_2 \cos\theta = 0.50$ [mm]. The applied in-plane stresses are $\bar{\sigma}_x = 50$ [MPa], $\bar{\sigma}_y = 50$ [MPa] and $\bar{\tau}_{xy} = 0$ [MPa] in orthogonal coordinates. The material properties used in the analysis are summarized in Table 1 and residual thermal stresses are neglected here. For the finite element calculation, the upper half of the laminate containing nine representative volumes is modeled subject to uniformly distributed forces except for crack surfaces. Stress distributions in the central representative volume are used for verification and the stresses are averaged along the thickness direction in one layer.

Analyses for three cases ($\theta = 30$, 45, 60 degrees) are performed and comparison of the stress distributions (transverse stress normal to fiber direction) in 90-degree layers obtained from the present analysis and finite element analysis are shown in

**Table 1.**
Material properties used in the analysis

| $E_L$ (GPa) | $E_T$ (GPa) | $\nu_{LT}$ | $\nu_{TT}$ | $G_{LT}$ (GPa) | $G_{TT}$ (GPa) |
|---|---|---|---|---|---|
| 147 | 8.3 | 0.352 | 0.45 | 4.7 | 2.87 |

Figs 4–6. The $x$ and $y$ axes denote the oblique coordinate and stress components in the orthogonal coordinates are used here. These stress components increase as approaching $x = \pm l_2$ on account of the existence of matrix cracks in $\theta$-ply. For all cases, results from two analyses are closely comparable. Although some deviation of stress distribution near the boundary is observed because of the use of average

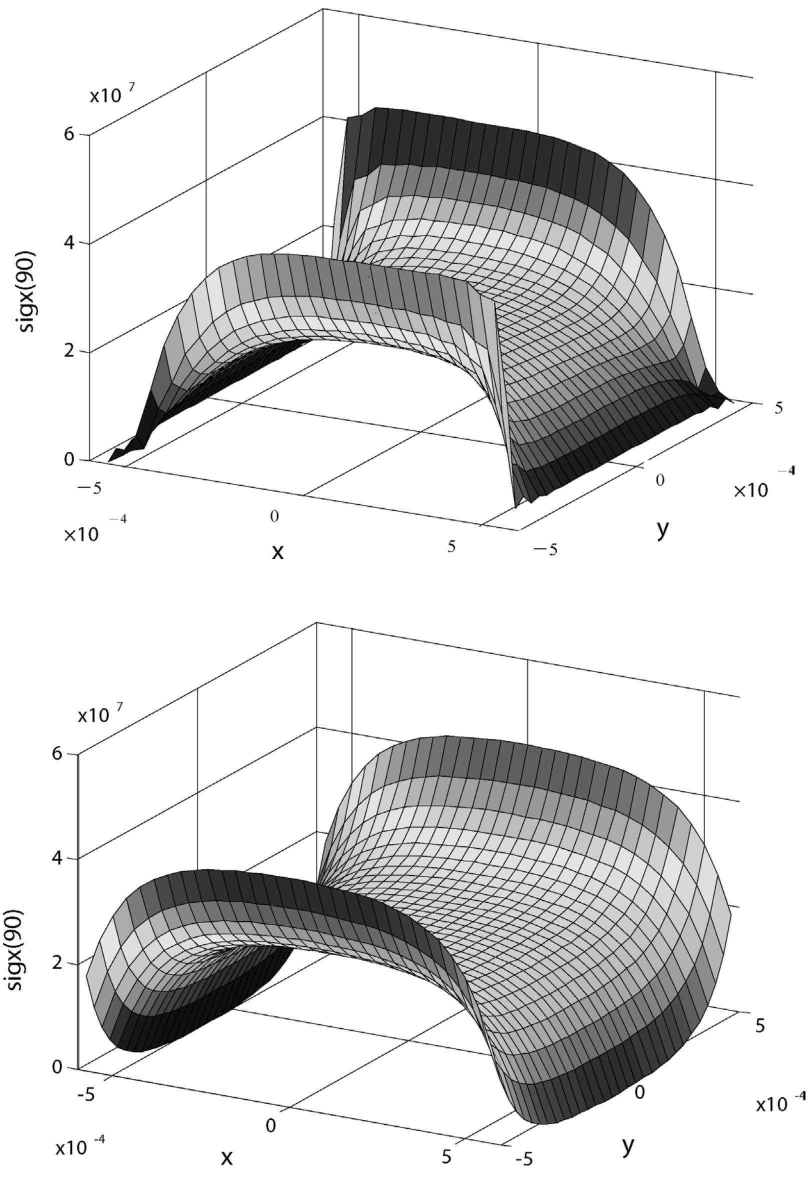



**Figure 4.** Transverse normal stress distribution in 90° layer of [30/90]s laminate subjected to bi-axial loadings: 2D shear-lag (upper) and FEM (lower).

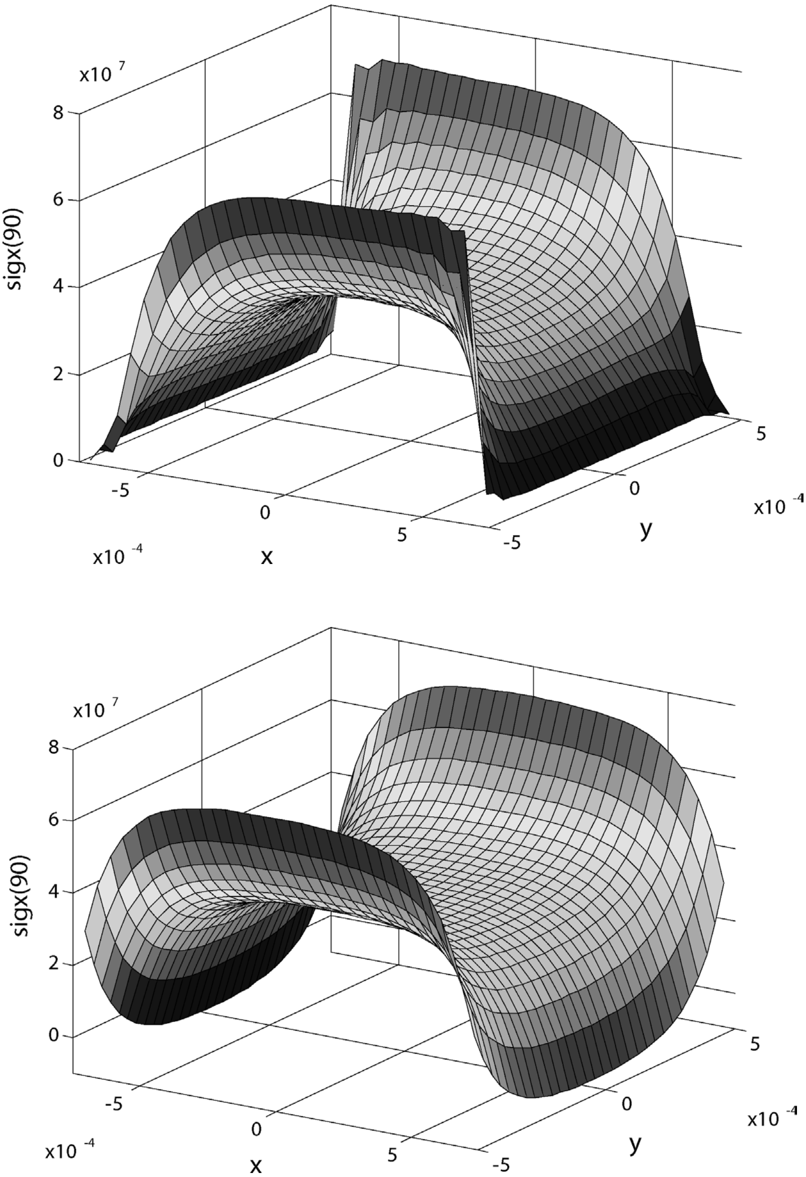



**Figure 5.** Transverse normal stress distribution in 90° layer of [45/90]s laminate subjected to bi-axial loadings: 2D shear-lag (upper) and FEM (lower).

boundary conditions in shear-lag analysis, fairly good correlation between the two is obtained. Cross-sectional views of the distributions of [45/90]s laminate are shown in Fig. 7. Effectiveness of the proposed simple analysis is also confirmed through this figure. It should be noted that this analytical solution is attained with little

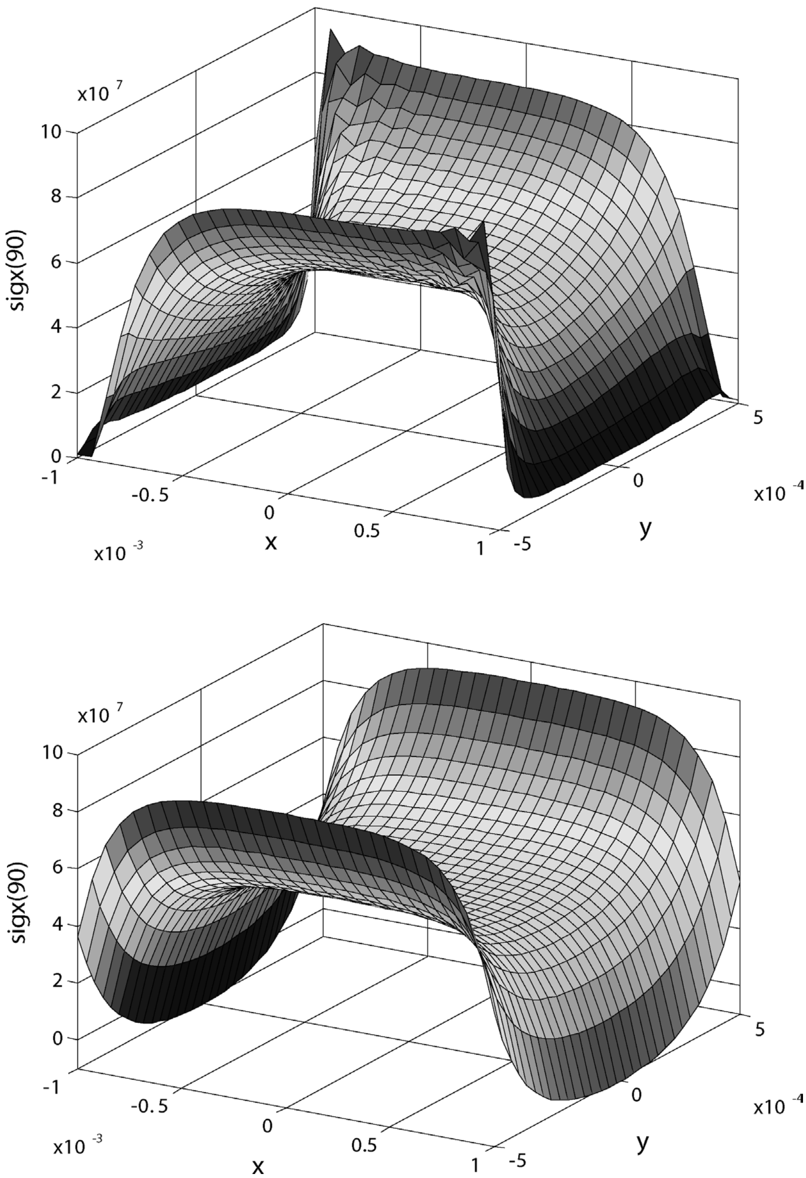



**Figure 6.** Transverse normal stress distribution in 90° layer of [60/90]s laminate subjected to bi-axial loadings: 2D shear-lag (upper) and FEM (lower).

more effort than the case of cross-ply cracks [23–27] because oblique problems are considered as orthogonal problems.

Stress analysis of $[-\theta/\theta/90]$s laminates containing $\theta$- and 90-ply matrix cracks is presented next. The crack configuration is identical to the previous example;

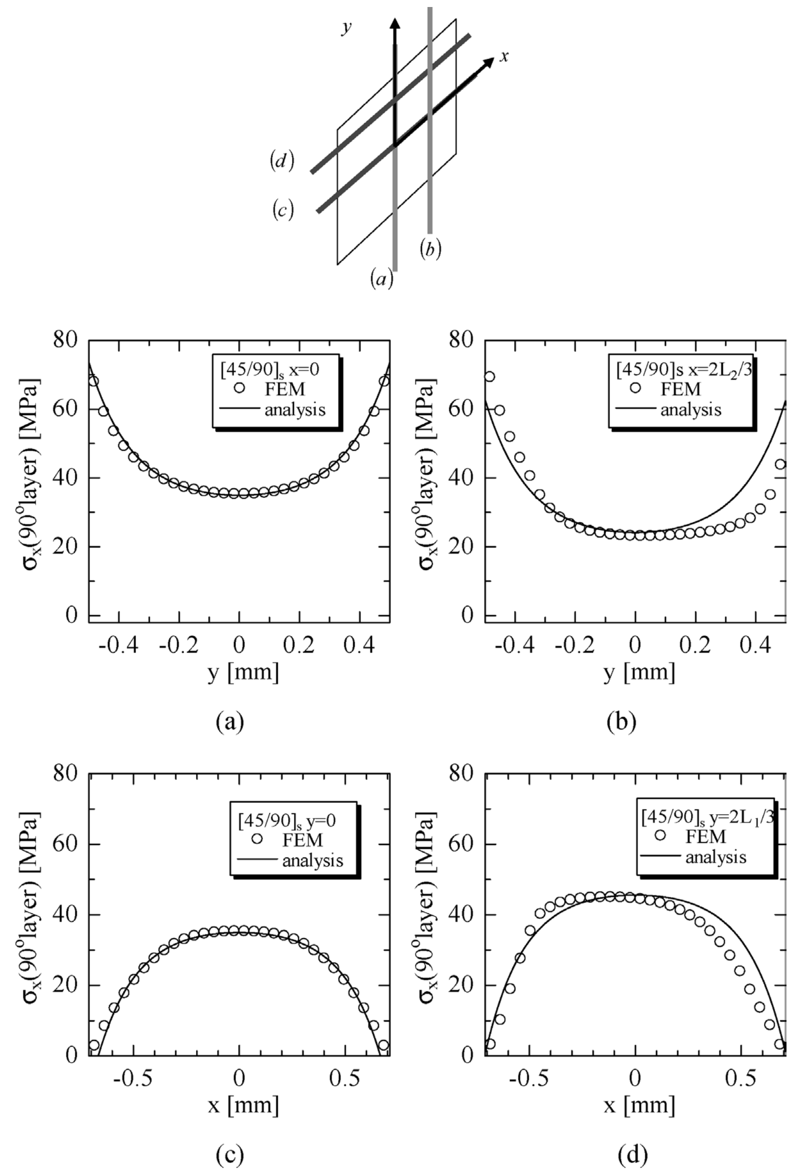
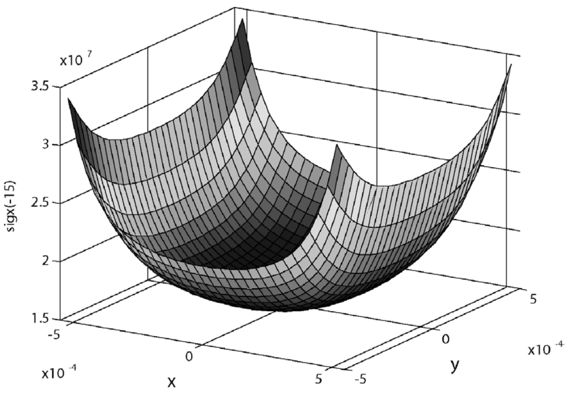
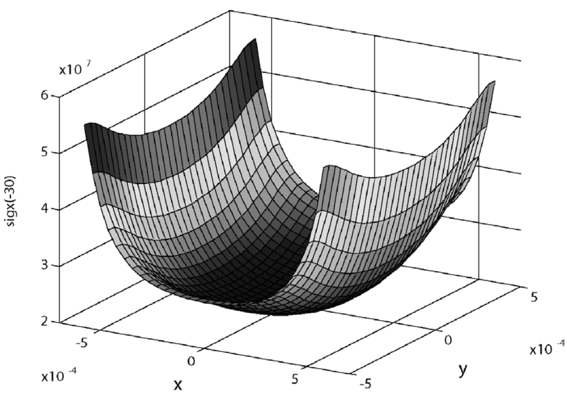



**Figure 7.** Cross-sectional distribution of transverse normal stress in 90° layer of [45/90]s laminate subjected to bi-axial loadings.

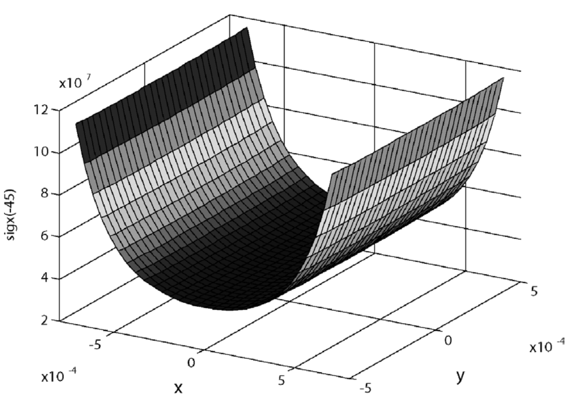
$l_1 = 0.50$ [mm] and $l_2 \cos\theta = 0.50$ [mm]. The applied in-plane strains are $\bar{\varepsilon}_x = 0.50\%$, $\bar{\varepsilon}_x = \bar{\gamma}_{xy} = 0$ in the orthogonal coordinates. The stress distributions (transverse stress normal to fiber direction) in the $-\theta$ ply ($\theta = 15, 30$ and $45$) are plotted in Fig. 8. Stress components increase at the vicinity of crossing regions of $\theta$- and 90-ply cracks. Thus, in this laminate configuration, matrix cracks in the




(a) [-15/15/90]s, -15ply




(b) [-30/30/90]s, -30ply




(c) [-45/45/90]s, -45ply

**Figure 8.** Transverse normal stress distribution in $-\theta°$ layer of $[-\theta/\theta/90]$s laminates with obliquely-crossed $\theta$- and 90-ply cracks: (a) $[-15/15/90]$s, (b) $[-30/30/90]$s, (c) $[-45/45/90]$s.

$-\theta$ ply are susceptible to initiation at the vicinity of the intersections of $\theta$- and 90-ply cracks.

## 5. CONCLUSIONS

Simple methodology for stress analysis of laminates with obliquely-crossed matrix cracks is presented with the introduction of oblique coordinate system along obliquely crossed crack surfaces. Covariant displacement/strain and contravariant force/stress components in conjunction with the associated constitutive equation are derived in relation to orthogonal coordinate components. It is identified that the obliquely-crossed crack problem with arbitrary oblique angles can be treated as an orthogonally-crossed crack problem, which ensures that obliquely-crossed cracked laminates can be regarded as orthogonally-crossed cracked laminates with appropriate properties.

This approach is combined with two-dimensional shear-lag analysis and analytical solutions of $[\theta_m/90_n]$s and $[S/\theta_m/90_n]$s laminates with matrix cracks in both $\theta$- and 90-plies under general in-plane loadings are obtained in terms of oblique coordinate components. Calculated stress distributions are verified with 3D finite element analyses and the effectiveness of the proposed method is confirmed. Multiple ply crack accumulation process is also investigated by calculating the stress distributions in the adjacent intact layer in the presence of obliquely-crossed cracks.

Because of the treatment of oblique problems as orthogonal problems, this analytical procedure has no more difficulties than the cross-ply cases [23–27]. The present formulation is based on the direct treatment of obliquely-crossed cracks as stress-free boundaries. Therefore, the synergistic effects between matrix cracks in two layers, which are included only in average sense in the homogenization method [19, 20], are incorporated in this analysis. The present method is expected to contribute to the clarification of the multi-ply damage mechanism in laminated structures.



## REFERENCES

1. K. W. Garrett and J. E. Bailey, Multiple transverse fracture in 90° cross-ply laminates of a glass fibre-reinforced polyester, *J. Mater. Sci.* **12**, 157–168 (1977).
2. D. L. Flaggs, Prediction of tensile matrix failure in composite laminates, *J. Compos. Mater.* **19**, 29–50 (1985).
3. S. L. Ogin, P. A. Smith and P. W. R. Beaumont, Matrix cracking and stiffness reduction during the fatigue of a [0/90]s GFRP laminate, *Compos. Sci. Technol.* **22**, 23–31 (1985).
4. Z. Hashin, Analysis of cracked laminates: a variational approach, *Mech. Mater.* **4**, 121–136 (1985).
5. J. A. Nairn, The strain energy release rate of composite microcracking: a variational approach, *J. Compos. Mater.* **23**, 1106–1129 (1989).

6. Y. M. Han, H. T. Hahn and R. B. Croman, A simplified analysis of transverse ply cracking in cross-ply laminates, *Compos. Sci. Technol.* **31**, 165–177 (1988).
7. S. C. Lim and C. S. Hong, Prediction of transverse cracking and stiffness reduction in cross-ply laminated composites, *J. Compos. Mater.* **23**, 695–713 (1989).
8. L. N. McCartney, Theory of stress transfer in a 0°-90°-0° cross-ply laminate containing a parallel array of transverse cracks, *J. Mech. Physics of Solids* **40**, 27–68 (1992).
9. G. A. Schoeppner and N. J. Pagano, Stress fields and energy release rates in cross-ply laminates, *Intern. J. Solids and Structures* **35**, 1025–1055 (1998).
10. R. Joffe and J. Varna, Analytical modeling of stiffness reduction in symmetric and balanced laminates due to cracks in 90° layers, *Compos. Sci. Technol.* **59**, 1641–1652 (1999).
11. J. A. Nairn and S. Hu, The formation and effect of outer-ply microcracks in cross-ply laminates: a variational approach, *Engng Fracture Mech.* **41**, 203–221 (1992).
12. J. A. Nairn, Microcracking, microcrack-induced delamination and longitudinal splitting of advanced composite structures, NASA CR-4472 (1992).
13. J. A. Nairn and S. Hu, Micromechanics of damage: a case study of matrix microcracking, in: *Damage Mechanics of Composite Materials*, R. Talreja (Ed.), pp. 187–243. Elsevier Science, Amsterdam (1994).
14. C. L. Tsai and I. M. Daniel, The behavior of cracked cross-ply composite laminates under shear loading, *Intern. J. Solids and Structures* **29**, 3251–3267 (1992).
15. L. N. McCartney, The prediction of cracking in biaxially loaded cross-ply laminates having brittle matrix, *Composites* **24**, 84–92 (1993).
16. L. N. McCartney, Model to predict effects of triaxial loading on ply cracking in general symmetric laminates, *Compos. Sci. Technol.* **60**, 2255–2279 (2000).
17. R. Talreja, Damage characterization by internal variables, in: *Damage Mechanics of Composite Materials*, R. Talreja (Ed.), pp. 53–78. Elsevier, Amsterdam (1994).
18. D. H. Allen, Damage evolution in laminates, in: *Damage Mechanics of Composite Materials*, R. Talreja (Ed.), pp. 79–114. Elsevier, Amsterdam (1994).
19. H. L. McManus and J. R. Maddocks, On microcracking in composite laminate under thermal and mechanical loading, *Polym. Polym. Compos.* **4**, 304–314 (1996).
20. C. H. Park and H. L. McManus, Thermally induced damage in composite laminates: predictive methodology and experimental investigation, *Compos. Sci. Technol.* **56**, 1209–1219 (1996).
21. Z. Hashin, Analysis of orthogonally cracked laminates under tension, *J. Appl. Mech.* **54**, 872–879 (1987).
22. J. Aboudi, S. W. Lee and C. T. Herakovich, Three-dimensional analysis of laminates with cross cracks, *J. Appl. Mech.* **55**, 389–397 (1988).
23. C. L. Tsai, I. M. Daniel and J. W. Lee, Progressive matrix cracking of crossply composite laminates under biaxial loading, in: *Microcracking-induced Damage in Composites*, G. J. Dvorak and D. C. Lagoudas (Eds), pp. 9–18. American Society of Mechanical Engineers, New York (1990).
24. C. L. Tsai and I. M. Daniel, The behavior of cracked crossply composite laminates under general in-plane loading, in: *Damage in Composite Materials*, G. Z. Voyiadjis (Ed.), pp. 51–66. Elsevier, Amsterdam (1993).
25. C. Henaff-Gardin, M. C. Lafarie-Frenot and D. Gamby, Doubly periodic matrix cracking in composite laminates. Part 1: general in-plane loading, *Composite Structures* **36**, 113–130 (1996).
26. C. Henaff-Gardin, M. C. Lafarie-Frenot and D. Gamby, Doubly periodic matrix cracking in composite laminates. Part 2: thermal biaxial loading, *Composite Structures* **36**, 131–140 (1996).
27. M. Kashtalyan and C. Soutis, Stiffness degradation in cross-ply laminates damaged by transverse cracking and splitting, *Composites (A)* **31**, 107–119 (2000).
28. Y. C. Fung, *Foundations of Solid Mechanics*. Prentice-Hall, Englewood Cliffs, NJ, USA (1965).
29. J. F. Caron and R. P. Carreira, Interface behavior in laminates with simplified model, *Compos. Sci. Technol.* **63**, 633–640 (2003).

**APPENDIX**

Each component of the stiffness matrix in equation (15) is written below.

$$[Q^{ijkl}] = \begin{bmatrix} Q_{11} & Q_{12} & Q_{13} & 0 & 0 & Q_{16} \\ & Q_{22} & Q_{23} & 0 & 0 & Q_{26} \\ & & Q_{33} & 0 & 0 & Q_{36} \\ & & & Q_{44} & Q_{45} & 0 \\ & sym & & & Q_{55} & 0 \\ & & & & & Q_{66} \end{bmatrix}, \tag{A.1}$$

$$\begin{aligned}
Q_{11} &= \bar{Q}_{11}\sec^4\theta, \\
Q_{12} &= \bar{Q}_{11}\sec^2\theta\tan^2\theta + \bar{Q}_{12}\sec^2\theta - 2\bar{Q}_{16}\sec^2\theta\tan\theta, \\
Q_{13} &= \bar{Q}_{13}\sec^2\theta, \\
Q_{16} &= -\bar{Q}_{11}\sec^3\theta\tan\theta + \bar{Q}_{16}\sec^3\theta, \\
Q_{22} &= \bar{Q}_{11}\tan^4\theta + 2(\bar{Q}_{12} + 2\bar{Q}_{66})\tan^2\theta + \bar{Q}_{22} \\
&\quad - 4\bar{Q}_{16}\tan^3\theta - 4\bar{Q}_{26}\tan\theta, \\
Q_{23} &= \bar{Q}_{23} + \bar{Q}_{13}\tan^2\theta - 2\bar{Q}_{36}\tan\theta, \\
Q_{26} &= -\bar{Q}_{11}\sec\theta\tan^3\theta - (\bar{Q}_{12} + 2\bar{Q}_{66})\sec\theta\tan\theta \\
&\quad + 3\bar{Q}_{16}\sec\theta\tan^2\theta + \bar{Q}_{26}\sec\theta, \\
Q_{33} &= \bar{Q}_{33}, \\
Q_{36} &= -\bar{Q}_{13}\sec\theta\tan\theta + \bar{Q}_{36}\sec\theta, \\
Q_{44} &= \bar{Q}_{44} - 2\bar{Q}_{45}\tan\theta + \bar{Q}_{55}\tan^2\theta, \\
Q_{45} &= \bar{Q}_{45}\sec\theta - \bar{Q}_{55}\sec\theta\tan\theta, \\
Q_{55} &= \bar{Q}_{55}\sec^2\theta, \\
Q_{66} &= \bar{Q}_{11}\sec^2\theta\tan^2\theta - 2\bar{Q}_{16}\sec^2\theta\tan\theta + \bar{Q}_{66}\sec^2\theta.
\end{aligned} \tag{A.2}$$